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Circular mixed hypergraphs II: The upper chromatic number

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Abstract

A mixed hypergraph is a triple $\mathscr{H} = (X, \mathscr{C}, \mathscr{D})$, where X is the vertex set and each of $\mathscr{C}$, $\mathscr{D}$ is a family of subsets of X, the $\mathscr{C}$-edges and $\mathscr{D}$-edges, respectively. A proper k-coloring of $\mathscr{H}$ is a mapping $c : X \to [k]$ such that each $\mathscr{C}$-edge has two vertices with a common color and each $\mathscr{D}$-edge has two vertices with distinct colors. A mixed hypergraph $\mathscr{H}$ is called *circular* if there exists a host cycle on the vertex set X such that every edge ($\mathscr{C}$- or $\mathscr{D}$-) induces a connected subgraph of this cycle.

We suggest a general procedure for coloring circular mixed hypergraphs and prove that if $\mathscr{H}$ is a reduced colorable circular mixed hypergraph with n vertices, upper chromatic number $\bar{\chi}$ and sieve number s, then

$$n - s - 2 \leqslant \bar{\chi} \leqslant n - s + 2.$$

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1. Introduction

In the classical theory of coloring graphs and hypergraphs [1,2], we ask for colorings of the vertices so that each edge requires at least two vertices of different colors, and ask for the minimum number of colors required. It is natural to ask the dual question to color the vertices so that each edge requires at least two vertices of the same color, and ask for the maximum number of colors needed. It is also natural to ask the combination of the above two questions [7,14–17,21].

In the present paper we deal with such a combination of constraints on colorings and use the terminology of [18]. Our paper [19] is continued (see also [20]).

A *mixed hypergraph* is a triple $\mathscr{H} = (X, \mathscr{C}, \mathscr{D})$, where X is the *vertex set* and each of $\mathscr{C}$, $\mathscr{D}$ is a family of subsets of X, the $\mathscr{C}$*-edges* and $\mathscr{D}$*-edges*, respectively. Each element of $\mathscr{C} \cup \mathscr{D}$ is of size at least 2. A proper k-coloring of a mixed hypergraph is a mapping from the vertex set to a set of k colors so that each $\mathscr{C}$-edge has two vertices with a Common color and each $\mathscr{D}$-edge has two vertices with $\mathscr{D}$ifferent colors. A mixed hypergraph is *k-colorable* (*uncolorable*) if it has

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a proper coloring with at most k colors (admits no proper colorings). A *strict* k-coloring is a proper coloring using all k colors. The minimum number of colors in a proper coloring of $\mathscr{H}$ is the *lower chromatic number* $\chi(\mathscr{H})$; the maximum number of colors in a strict coloring is the *upper chromatic number* $\bar{\chi}(\mathscr{H})$. We use $c(x)$ for the color of the vertex x in the coloring c.

If $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ is a mixed hypergraph, then the subhypergraph *induced* by $X'\subseteq X$ is the mixed hypergraph $\mathscr{H}'=(X',\mathscr{C}',\mathscr{D}')$ defined by $\mathscr{C}'=\{C\in\mathscr{C}:C\subseteq X'\}$ and $\mathscr{D}'=\{D\in\mathscr{D}:D\subseteq X'\}$. A mixed hypergraph $\mathscr{H}'=(X',\mathscr{C}',\mathscr{D}')$ is a *mixed subhypergraph* of $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ if $X'\subseteq X, \mathscr{C}'\subseteq\mathscr{C}, \mathscr{D}'\subseteq\mathscr{D}$. For the last we use the notation $\mathscr{H}'\subseteq\mathscr{H}$. For short, sometimes we write $(X,\mathscr{D})$ or $\mathscr{H}_{\mathscr{D}}$ instead of $\mathscr{H}=(X,\emptyset,\mathscr{D})$, and write $(X,\mathscr{C})$ or $\mathscr{H}_{\mathscr{C}}$ instead of $\mathscr{H}=(X,\mathscr{C},\emptyset)$, keeping in mind that the respective coloring restrictions are fulfilled.

A mixed hypergraph is *reduced* if it contains no included $\mathscr{C}$-edges and no included $\mathscr{D}$-edges, and moreover, the size of each $\mathscr{C}$-edge is at least 3, and the size of each $\mathscr{D}$-edge is at least 2. As it follows from the splitting–contraction algorithm [17], the coloring properties of arbitrary mixed hypergraphs can be derived from the respective reduced mixed hypergraph. Therefore, without loss of generality, throughout the paper we consider reduced mixed hypergraphs.

In a mixed hypergraph $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ the subfamily of $\mathscr{C}$-edges $\Sigma\subseteq\mathscr{C}$ is a *sieve* [3], if for every pair of vertices $x,y\in X$ and every pair of different $\mathscr{C}$-edges $C,C'\in\Sigma$ the following implication holds:

$$\{x,y\}\in C\cap C'\Rightarrow\{x,y\}\in\mathscr{D}.$$

The maximum cardinality of a sieve of a hypergraph $\mathscr{H}$ is the sieve-number $s(\mathscr{H})$.

A mixed hypergraph $\mathscr{H}$ is called an *interval mixed hypergraph* [3] if there exists a linear ordering of its vertices such that every edge ($\mathscr{C}$- or $\mathscr{D}$-) induces an interval in that ordering. Interval mixed hypergraphs have been introduced and investigated in [3], where it was shown in particular that for a colorable interval mixed hypergraph $\chi(\mathscr{H})\leqslant 2$ and $\bar{\chi}(\mathscr{H})=n-s(\mathscr{H})$.

Introduced in [16,17], the theory of mixed hypergraphs is growing rapidly and represents an area with many possible applications. As models they may be applied in list-free modelling of list-colorings of graphs [14], integer programming [14,18], investigating the coloring properties of block designs [4–6,11,12], resource allocation, Data Base Management, parallel computing, scheduling of systems of power supplies, in the study of heredity in populations with sexual reproduction [18] where the problems have combinatorial nature.

There are several classes of mixed hypergraphs which have been introduced and investigated recently; among them are mixed hypertrees [9,13,14,17], mixed hyperkakti [10], perfect [8,17,18], mixed interval [3], uncolorable [14], uniquely colorable [15], pseudo-chordal [21], and some other mixed hypergraphs. They represent the generalizations of graphs, hypergraphs and colorings from different points of view. Mixed interval hypergraphs and mixed hypertrees represent the generalizations based on the underlying host graph (for a hypergraph the host graph is the graph on the same vertex set and such that every hyperedge induces a connected subgraph of the host graph). Namely in those cases the host graphs are paths and trees respectively, i.e. graphs without cycles. In this sense they express nothing else than the coloring properties of generalized paths and trees with respect to their connected subgraphs.

Therefore, the main goal of this paper is to do the next step and investigate what happens with the colorings in the language of mixed hypergraphs if the host graph is a simple cycle. And the main conclusion will be that such cyclic ordering of the vertices of mixed hypergraph allows the efficient determination of both chromatic numbers and the optimal colorings. Moreover, as we will see, the value of the upper chromatic number is not far from $n-s$, the exact value of the upper chromatic number for interval mixed hypergraphs.

Definition 1.1. A mixed hypergraph $\mathscr{H}$ is called *circular* if there exists a host cycle on the vertex set X such that every $\mathscr{C}$-edge and every $\mathscr{D}$-edge induces a connected subgraph of the host cycle.

In other words, for circular mixed hypergraph there exists always a circular ordering of the vertex set X, say, $X=\{x_1,x_2,\dots,x_n,x_1\}$ such that every edge ($\mathscr{C}$- or $\mathscr{D}$-) induces an interval in this ordering. Notice that any circular mixed hypergraph can be made reduced (by elimination of edges containing other edges of the same type and contraction of $\mathscr{C}$-edges of size 2) without leaving the class of circular mixed hypergraphs.

In [19] we investigated the lower chromatic number, colorability and unique colorability of circular mixed hypergraphs. As it easily follows from [19], for any colorable circular mixed hypergraph with n vertices the lower chromatic number $\chi=3$ if and only if n is odd and $\mathscr{D}$ is a classic style, otherwise $\chi\leqslant 2$. In the present paper we determine the values of the upper chromatic number $\bar{\chi}$.

The paper is organized as follows. In Section 2, we discuss an upper bound for the upper chromatic number. In Section 3, we introduce the so-called coloring procedure $\mathscr{M}$. We use it to obtain the lower bounds for the upper chromatic number in Section 4. At last, in Section 5, we formulate our main result: if $\mathscr{H}$ is a reduced colorable circular mixed hypergraph with the upper chromatic number $\bar{\chi}(\mathscr{H})$, then

$$n - s - 2 \leqslant \bar{\chi}(\mathscr{H}) \leqslant n - s + 2,$$

where s is the sieve number.

On the way to this main conclusion, the classes of circular mixed hypergraphs $\mathscr{H}$ with $\bar{\chi}(\mathscr{H}) = n - s + 2$ and $\bar{\chi}(\mathscr{H}) = n - s - 2$ are characterized. In addition, it is shown that all intermediate values of the upper chromatic number between $n - s - 2$ and $n - s + 2$ are attained for some special cases.

2. An upper bound for the upper chromatic number

Let us strongly obey the agreement that all the vertices are ordered on the host cycle and in all expressions like '(u, v)-path' or '$\mathscr{D}$-distance between u and v' we mean the host-path or the number of vertices minus 1 in this path, respectively, when starting at u and ending at v and going according the given cyclic ordering. Moreover, for the vertex $u \in X$ the notation $u^{--}, u^{-}, u^{+}, u^{++}, \ldots$ or, equivalently, $u^{-2}, u^{-1}, u^{1}, u^{2}, \ldots$ will always mean the second predecessor, the first predecessor, the first successor, the second successor, ... of u in the cyclic ordering. In this way, any sequence of vertices always assumes the ordering along the host cycle. Since no edge contains another edge of the same type, any set of edges of the same type by default is ordered along the host cycle. Such an ordering is clearly induced by the first vertex of every edge.

Definition 2.1. Let $S = (x_0, C_1, x_1, C_2, x_2, \ldots, x_{t-2}, C_{t-1}, x_{t-1}, C_t, x_t)$ be a sequence of vertices $x_0, \ldots, x_t$ and $\mathscr{C}$-edges $C_1, \ldots, C_t$ of size 3 such that x_{i-1} is the first vertex and x_i is the last vertex of C_i (the edges C_i need not be disjoint), $i = 1, \ldots, t$. The sequence S is called an *x_0x_t-triple-chain* if $x_t \neq x_0$ and is called a *triple-cycle* or a *closed triple-chain* if $x_t = x_0$. A u-triple-cycle is a triple-cycle through the vertex u.

An x_0x_t-triple-chain may pass (transport) the color from the vertex x_0 to the vertex x_t. Namely, if every two consecutive vertices of the host path corresponding to S form a $\mathscr{D}$-edge, then necessarily $c(x_0) = c(x_1) = \cdots = c(x_t)$, though intermediate vertices may have arbitrary colors different from $c(x_0)$. Such a triple-chain is said to be *strong*.

Definition 2.2. Let $\mathscr{H} = (X, \mathscr{C}, \mathscr{D})$ be a circular mixed hypergraph. If every k consecutive vertices of X form a $\mathscr{C}$-edge, then we denote $\mathscr{C} = \mathscr{C}_k$. Similarly, if every l consecutive vertices of X form a $\mathscr{D}$-edge, then we denote $\mathscr{D} = \mathscr{D}_l$. The circular mixed hypergraph $\mathscr{H} = (X, \mathscr{C}_k, \mathscr{D}_l)$ is called *(k, l)-uniform* and is denoted by $\mathscr{KC}(n; k, l)$, where $n := |X|$. A (3,2)-uniform circular mixed hypergraph $\mathscr{H} = (X, \mathscr{C}_3, \mathscr{D}_2) = \mathscr{KC}(n; 3, 2)$ is called a *complete circular mixed hypergraph*.

Thus the mixed hypergraph $(X, \emptyset, \mathscr{D}_2)$ is a simple classical cycle on n vertices.

Definition 2.3. For every odd $n \geqslant 3$ we define a circular mixed hypergraph $\mathscr{F}_n$ such that $\mathscr{F}_n = (X, \mathscr{C}, \mathscr{D}_2)$ where $(X, \emptyset, \mathscr{D}_2)$ is a classical cycle and $(X, \mathscr{C}, \emptyset)$ is a triple-chain which is minimal with respect to the property that the first and the last vertices of it are neighbors on the host cycle.

The number of $\mathscr{C}$-edges and $\mathscr{D}$-edges of $\mathscr{F}_n$ is $(n - 1)/2$ or n, respectively. $\mathscr{F}_3$ consists of a $\mathscr{D}$-triangle and of one $\mathscr{C}$-edge coinciding with X.

Sieves and sieve numbers play an important role in estimation of the upper chromatic number. A sieve is a collection Σ of $\mathscr{C}$-edges so that the intersection $C \cap C'$ of any two different $\mathscr{C}$-edges C, C' form a complete $\mathscr{D}$-graph, i.e., any two different vertices of $C \cap C'$ form a $\mathscr{D}$-edge of size 2. Note that two different $\mathscr{C}$-edges C, C' cannot be colored properly by coloring two vertices of $C \cap C'$ with the same color.

In a circular mixed hypergraph, two $\mathscr{C}$-edges of a sieve have an empty intersection or an intersection consisting of one vertex or of two vertices joint by a $\mathscr{D}$-edge of size 2.

Theorem 2.1. *Let $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ be a reduced colorable circular mixed hypergraph with n vertices and sieve number s. Then $\bar{\chi}(\mathscr{H})\leqslant n-s$, if $0\leqslant s\leqslant 2$; $\bar{\chi}(\mathscr{H})\leqslant n-s+1$, if $s=3$; and $\bar{\chi}(\mathscr{H})\leqslant n-s+2$, if $s\geqslant 4$. The upper bound is sharp for $0\leqslant s\leqslant 3$.*

Proof. If $s=0$ then $\mathscr{C}=\emptyset$ and $\bar{\chi}(\mathscr{H})=n=n-s$. If $s=1$ then $\mathscr{C}\neq\emptyset$, at least two vertices have the same color and $\bar{\chi}(\mathscr{H})\leqslant n-1=n-s$. If $s=2$ then in each of the two $\mathscr{C}$-edges of a sieve at least two vertices have the same color and $\bar{\chi}(\mathscr{H})\leqslant n-2=n-s$. If $s=3$ then a subsieve of two $\mathscr{C}$-edges causes $\bar{\chi}(\mathscr{H})\leqslant n-2=n-s+1$.

Next let $s\geqslant 4$. Let $\Sigma=\{C_1,C_2,\ldots,C_s\}$ be a maximum sieve of a colorable circular mixed hypergraph with n vertices, where $C_1,C_2,\ldots,C_s$ are in this cyclic order on the host cycle. Let $\{x_0^1,x_1^1,x_2^1,\ldots\}$ be the vertex set of C_1 appearing in this cyclic order around the host cycle. Obviously, $x_0^1\notin C_{s-2}\cap C_1$. Hence in any proper coloring of $(X,\Sigma,\mathscr{D})$ the $\mathscr{C}$-edge C_i has two vertices of the same color, which do not belong both to C_j, for all $1\leqslant i,\ j\leqslant s-2,\ j\neq i$. Hence each $\mathscr{C}$-edge C_i causes a "repetition" of colors, and at least $s-2$ colors are "repeated". Therefore, $\bar{\chi}(X,\Sigma,\mathscr{D})\leqslant n-s+2$, what implies $\bar{\chi}(\mathscr{H})\leqslant n-s+2$. $\square$

The next theorem shows that the bound of Theorem 2.1 is only sharp for even $n\geqslant 6$.

Theorem 2.2. *Let $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ be a reduced colorable circular mixed hypergraph with n vertices and sieve number $s\geqslant 4$. Then*

$$\bar{\chi}(\mathscr{H})=n-s+2$$

if and only if n is even, $n\geqslant 6$, and $\mathscr{H}=\mathscr{K}\mathscr{C}(n;3,2)$. Moreover, for all such cases, $n-s+2=2$.

Proof. Let $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ be a circular mixed hypergraph with $\bar{\chi}(\mathscr{H})=n-s+2$ and a maximum sieve $\Sigma=\{C_1,C_2,\ldots,C_s\}$ arranged in this cyclic order around the host cycle. If $x_0^1\notin C_{s-1}\cap C_1$ then the proof of Theorem 2.1 implies that $\bar{\chi}(\mathscr{H})\leqslant n-s+1$. Next let $x_0^1\in C_{s-1}\cap C_1$. The $\mathscr{C}$-edge C_s has a proper intersection with both C_{s-1} and C_1. Since $C_{s-1},C_s,C_1\in\Sigma$ the size of C_s is 3. Therefore, $C_s=\{x_0^1,x_1^1,x_2^1\}$, and the definition of a sieve implies that $\{x_0^1,x_1^1\},\{x_1^1,x_2^1\}\in\mathscr{D}$. This reasoning may be repeated for every $\mathscr{C}$-edge from Σ. Therefore, all $\mathscr{C}$-edges of Σ have size 3, each two consecutive $\mathscr{C}$-edges of Σ meet in two vertices and any two consecutive vertices of $\mathscr{H}$ are joined by a $\mathscr{D}$-edge of size 2. For odd n the circular mixed hypergraph $\mathscr{H}=\mathscr{K}\mathscr{C}(n;3,2)$ is uncolorable. For $n=4$, We have $s=2$ and $\bar{\chi}(\mathscr{K}\mathscr{C}(4;3,2))=2\neq n-s+2$. Hence, the assertion of the theorem follows. $\square$

The proofs of Theorems 2.1 and 2.2 imply

Corollary 2.1. *Let $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ be a circular mixed hypergraph with n vertices and sieve number s, $s\geqslant 3$. Let $\Sigma=\{C_1,C_2,\ldots,C_s\}$ be a sieve of $\mathscr{H}$. It follows*:

if $C_s\cap C_1=\emptyset$ then $\bar{\chi}(\mathscr{H})\leqslant n-s$, and
if $C_{s-1}\cap C_1=\emptyset$ then $\bar{\chi}(\mathscr{H})\leqslant n-s+1$,
otherwise $\bar{\chi}(\mathscr{H})\leqslant n-s+2$.

Now we discuss the following three special cases which will be used in the next sections.

Theorem 2.3. *Let $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ be a reduced circular mixed hypergraph with n vertices, $n\geqslant 6$ even, $\mathscr{C}\subseteq\mathscr{C}_3$, $\mathscr{D}=\mathscr{D}_2$. Then the following holds*:

If $\mathscr{C}=\mathscr{C}_3$ then $\bar{\chi}(\mathscr{H})=n-s+2=2$.
If $\mathscr{C}\neq\mathscr{C}_3$ and $(X,\mathscr{C})$ contains a triple-cycle, then $\bar{\chi}(\mathscr{H})=n-s+1$.
If $(X,\mathscr{C})$ contains no triple-cycle, then $\bar{\chi}(\mathscr{H})=n-s$.

Proof. Since $\mathscr{D}=\mathscr{D}_2$ and $n\geqslant 6$, the whole set $\mathscr{C}$ forms a maximum sieve of $\mathscr{H}$, i.e. $|\mathscr{C}|=s(\mathscr{H})$. Firstly, we remark that in each proper coloring of $\mathscr{H}$ the vertices of each (strong) triple-chain and each (strong) triple-cycle have the same color. If $\mathscr{C}=\mathscr{C}_3$ then $\mathscr{H}$ contains two triple-cycles and therefore, $\bar{\chi}(\mathscr{H})=2$.

Next let $\mathscr{C} \neq \mathscr{C}_3$, i.e., $k\,\mathscr{C}$-edges of the complete circular mixed hypergraph $(X, \mathscr{C}_3, \mathscr{D}_2) \cong \mathscr{K}\mathscr{C}(n; 3, 2)$ are missing, $k := n - s \geqslant 1$.

Case 1. Let $\mathscr{H}$ contain one triple-cycle.

Then $\mathscr{H}$ can be obtained from the complete circular mixed hypergraph $\mathscr{K}\mathscr{C}(n; 3, 2)$ by deleting from one triple-cycle $k\,\mathscr{C}$-edges. Then $\mathscr{H}$ consists of one strong triple-cycle and k maximal strong triple-chains. Thus $\mathscr{H}$ is covered by $k + 1$ vertex disjoint (closed or not closed) triple-chains. In a proper coloring of $\mathscr{H}$ all vertices of a strong triple-chain have the same color. The maximum possible number of colors in a proper coloring is obtained by coloring the $k + 1$ triple-chains by $k + 1$ different colors. Hence $\bar{\chi}(\mathscr{H}) = k + 1 = n - s + 1$.

Case 2. Let $\mathscr{H}$ contain no triple-cycle.

Then $\mathscr{H}$ can be obtained from the complete circular mixed hypergraph $\mathscr{K}\mathscr{C}(n; 3, 2)$ by deleting k_1 $\mathscr{C}$-edges from one triple-cycle and k_2 $\mathscr{C}$-edges from the other triple-cycle, $k_1 \geqslant 1$, $k_2 \geqslant 1$, $k_1 + k_2 = k$. Thus $\mathscr{H}$ is covered by k vertex disjoint triple-chains. The maximum possible number of colors in a proper coloring is again obtained by coloring the k triple-chains by k different colors. Hence $\bar{\chi}(\mathscr{H}) = k = n - s$. □

Theorem 2.4. *Let $\mathscr{H} = (X, \mathscr{C}, \mathscr{D})$ be a reduced circular mixed hypergraph with n vertices, n odd, $\mathscr{C} \subseteq \mathscr{C}_3$, $\mathscr{D} = \mathscr{D}_2$. If $\mathscr{F}_n \subseteq \mathscr{H}$ then $\mathscr{H}$ is uncolorable. Otherwise $\mathscr{H}$ has a coloring with $\bar{\chi}(\mathscr{H}) = n - s$ colors.*

Proof. *Case* 1. $\mathscr{F}_n \subseteq \mathscr{H}$. Then the vertices of X can be denoted by $x_0, x_1, \ldots, x_{n-1}$ in a cyclic ordering such that

$$\mathscr{F}_n = x_0\mathscr{C}_1x_2\mathscr{C}_2x_4 \ldots x_{n-3}\mathscr{C}_tx_{n-1},$$

where $t = (n - 1)/2$. In each proper coloring of $\mathscr{H}$ the vertices $x_0, x_2, x_4, \ldots, x_{n-3}, x_{n-1}$ have the same color. This leads to a contradiction with $(x_{n-1}, x_0) \in \mathscr{D}$. Consequently, $\mathscr{H}$ has no proper colorings.

Case 2. $\mathscr{F}_n \nsubseteq \mathscr{H}$. Let the family of $\mathscr{C}$-edges form k maximal triple-chains, $k \geqslant 0$. First, observe that each $\mathscr{C}$-edge is in maximum sieve, so, $s(\mathscr{H}) = |\mathscr{C}|$. Now, following the ordering on the host cycle, let us sequentially eliminate the $\mathscr{C}$-edges of the first triple-chain, then the $\mathscr{C}$-edges of the second triple-chain, and so on until no $\mathscr{C}$-edges are left. Fix this ordering of the $\mathscr{C}$-edges. Remaining mixed hypergraph is just $(X, \mathscr{D}_2)$ with $s = 0$ and $\bar{\chi} = n = n - s$. Now we start adding the $\mathscr{C}$-edges of $\mathscr{H}$ in the ordering fixed above, step by step. Observe that on each such step, either we continue a triple-chain or start a new one, we lose one color in a maximum coloring and add one $\mathscr{C}$-edge to the maximum sieve. Therefore, on each step the equality $\bar{\chi} + s = n$ remains unchanged. Since $\mathscr{F}_n \nsubseteq \mathscr{H}$, we can continue this procedure until we arrive to $\mathscr{H}$ with $\bar{\chi}(\mathscr{H}) = n - s$. □

Theorem 2.5. *Let $\mathscr{H} = (X, \mathscr{C}, \mathscr{D})$ be a reduced circular mixed hypergraph with n vertices, $|C| \geqslant 5$ for all $C \in \mathscr{C}$. Then*

$$\bar{\chi}(\mathscr{H}) = n - s + 1$$

if and only if $\mathscr{C} = \Sigma \cup \mathscr{C}'$, $\Sigma \cap \mathscr{C}' = \emptyset$; where

(i) *$\Sigma = \{C_1, C_2, \ldots, C_s\}$ is a maximum sieve such that $1 \leqslant |C_i \cap C_{i+1}| \leqslant 2$ (and $C_i \cap C_{i+2} = \emptyset$) for all $1 \leqslant i \leqslant s$, $s \geqslant 3$ (indices mod s), and*

(ii) *each $\mathscr{C}$-edge $C \in \mathscr{C}'$ has the property: there exist two $\mathscr{C}$-edges $C_i, C_{i+1} \in \Sigma$ with two common vertices, say u and u^+, such that either $u^+ \notin C$ and $C_i \backslash \{u^+\} \subset C$ or $u \notin C$ and $C_{i+1} \backslash \{u\} \subset C$, and there is no other $\mathscr{C}$-edge of $\mathscr{C}'$ containing precisely one of the vertices u and u^+.*

Proof. $\Rightarrow$ Let $\mathscr{H} = (X, \mathscr{C}, \mathscr{D})$ be a circular mixed hypergraph such that $\bar{\chi}(\mathscr{H}) = n - s + 1$. Let $\Sigma = \{C_1, C_2, \ldots, C_s\}$ be a maximum sieve of $\mathscr{H}$.

Since $|C| \geqslant 5$ for any $C \in \mathscr{C}$, the intersection $C_i \cap C_{i+2} = \emptyset$ for all $1 \leqslant i \leqslant s$ (indices mod s). Note that $C_i \cap C_{i+2} \neq \emptyset$ would imply that C_{i+1} is a $\mathscr{C}$-edge of size 3.

Suppose $|C_i \cap C_{i+1} = \emptyset|$ for some $1 \leqslant i \leqslant s$. W.l.o.g. let $i = s$, i.e., $C_s \cap C_1 = \emptyset$. Corollary 2.1 implies the upper chromatic number $\bar{\chi}(\mathscr{H}) \leqslant n - s$. This contradiction shows that $1 \leqslant |C_i \cap C_{i+1}| \leqslant 2$ for all $1 \leqslant i \leqslant s$ (indices mod s). Hence $s \geqslant 3$ and $n \geqslant 9$.

Let c be a strict coloring of $\mathscr{H}$ with $n - s + 1$ colors. Corollary 2.1 implies that

$$\bar{\chi}(X, \Sigma \backslash \{C_s\}, \mathscr{D}) \leqslant n - s + 1.$$

It follows that

$$n-s+1=\bar{\chi}(\mathscr{H})\leqslant\bar{\chi}(X,\Sigma\backslash\{C_s\},\mathscr{D})\leqslant n-s+1.$$

Hence

$$\bar{\chi}(X,\Sigma\backslash\{C_s\},\mathscr{D})=n-s+1,$$

and in the coloring c each pair of vertices of the same color belongs to $C_1\cup C_2\cup\cdots\cup C_{s-1}$.

In c the $\mathscr{C}$-edge C_s is properly colored, that is C_s contains two vertices u, v of the same color which belong to $C_1\cup C_2\cup\cdots\cup C_{s-1}$. Hence $u, v\in C_{s-1}\cup C_1$.

Obviously, the vertices of $C_i\cap C_{i+1}$ have different colors. Consequently, one of u, v belongs to C_{s-1}, say, u, and the other $v\in C_1$.

Repeating this arguments for arbitrary i we arrive at the conclusion that each of the intersections $C_i\cap C_{i+1}$ and $C_{i+1}\cap C_{i+2}$ contains a vertex v_i and v_{i+1}, respectively, with $c(v_i)=c(v_{i+1})$. This implies $c(v_1)=c(v_2)=\cdots=c(v_s)$, and the vertex set $X\backslash\{v_1,\ldots,v_s\}$ is polychromatic. Since every $\mathscr{C}$-edge $C\in\mathscr{C}'$ must have at least two vertices of the same color, the assertion (ii) can easily be derived.

$\Leftarrow$ There is an obvious coloring of $\mathscr{H}$ with $n-s+1$ colors.

Color in each intersection $C_i\cap C_{i+1}$ precisely one vertex with 1, say the vertex v_i. If $C_i\cap C_{i+1}=\{u\}$ then color $v_i:=u$ with 1. Next let $C_i\cap C_{i+1}=\{u,u^+\}$. If there is a $\mathscr{C}$-edge $C\in\mathscr{C}\backslash\Sigma$ containing u or u^+ then color $v_i:=u$ or $v_i:=u^+$ with 1, respectively. If there is no such $\mathscr{C}$-edge then color an arbitrary vertex $v_i\in C_i\cap C_{i+1}=\{u,u^+\}$ with 1. Finally color the vertices of $X\backslash\{v_1,\ldots,v_s\}$ pairwise differently with $2,\ldots,n-s+1$. Easily can be proved that this coloring is proper. So a strict $(n-s+1)$-coloring of $\mathscr{H}$ is obtained. □

3. The procedure $\mathscr{M}$ and its properties

Here a method is presented to construct a sieve Σ of a circular mixed hypergraph $\mathscr{H}$ through a given $\mathscr{C}$-edge C_1 and coloring it almost properly, where Σ is maximum among all sieves of $\mathscr{H}$ containing C_1. In a circular mixed hypergraph the intersection of two different $\mathscr{C}$-edges of a sieve can only be empty, a common vertex, or two common vertices forming a $\mathscr{D}$-edge. Hence the intersection of two arbitrary $\mathscr{C}$-edges is called *good* if and only if this intersection is empty, or a single vertex, or two vertices forming a $\mathscr{D}$-edge. Otherwise, the intersection of two arbitrary $\mathscr{C}$-edges is called *bad*.

Coloring procedure $\mathscr{M}$. Let $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$ be a circular mixed hypergraph and C_1 one of its $\mathscr{C}$-edges.

1. *Construction of a maximal sieve* Σ *through* C_1. Let at the beginning $\Sigma=\{C_1\}$. Choose the $\mathscr{C}$-edge C_2 nearest to C_1, i.e., among all $\mathscr{C}$-edges having a "good intersection" with C_1 let C_2 have smallest distance from C_1 (measured in the cyclic order of the host cycle). Choose C_3 nearest to C_2, etc. Thus a maximal sieve $\Sigma=\{C_1,C_2,\ldots,C_t\}$ is obtained so that in the next step no new C_{t+1} can be found.

2. *Assigning colors to some vertices of* Σ. The vertices of the $\mathscr{C}$-edge C_i with k_i vertices are denoted by $x_1^i, x_2^i,\ldots,x_{k_i}^i$ according to the cyclic order of the host cycle. Next we assign colors to some vertices of the $\mathscr{C}$-edges of Σ in the following way, where the coloring is denoted by c.

To $x_{k_1}^1$ the color 1 is assigned. The color 1 is also assigned to $x_{k_1-1}^1$ if $(x_{k_1-1}^1,x_{k_1}^1)\notin\mathscr{D}$ and to $x_{k_1-2}^1$ if $(x_{k_1-1}^1,x_{k_1}^1)\in\mathscr{D}$.

If some vertices of $C_1,\ldots,C_{i-1}$ are already colored with the colors, say $1,2,\ldots,\alpha$, then one or two vertices of C_i are colored in the following way:

if $(x_{k_i-1}^i,x_{k_i}^i)\notin\mathscr{D}$ and $x_{k_i-1}^i$ is already colored then put $c(x_{k_i}^i)=c(x_{k_i-1}^i)$;

if $(x_{k_{i-1}}^i,x_{k_i}^i)\notin\mathscr{D}$ and $x_{k_i-1}^i$ is not already colored then assign a new color $\alpha+1$ to both $x_{k_i-1}^i$ and $x_{k_i}^i$;

if $(x_{k_i-1}^i,x_{k_i}^i)\in\mathscr{D}$ and $x_{k_i-2}^i$ is already colored then put $c(x_{k_i}^i)=c(x_{k_i-2}^i)$;

if $(x_{k_i-1}^i,x_{k_i}^i)\in\mathscr{D}$ and $x_{k_i-2}^i$ is not already colored then assign a new color $\alpha+1$ to $x_{k_i-2}^i$ and $x_{k_i}^i$.

3. *Fixing the vertices to which two colors are assigned.* In 2, we start coloring with the vertex $x^1_{k_1} \in C_1$, proceed along the host cycle and end with coloring of the vertices of C_t. If $C_t \cap C_1 \neq \emptyset$ or even $C_{t-1} \cap C_1 \neq \emptyset$ then it may occur that to some of the two vertices x^1_1, x^1_2 two colors are assigned: 1 at the beginning of 2. and α or $\alpha + 1$ at the end. In this case, we keep two colors assigned to each of these vertices until some recoloring is made during the proofs. (If both vertices are colored twice then C_1 is a triple $C_1 = \{x^1_1, x^1_2, x^1_3\}$ and $\{x^1_1, x^1_2\}, \{x^1_2, x^1_3\} \in \mathscr{D}$; this possibility will be avoided in the following proof, so we have only to investigate the case that one vertex is colored twice, namely, x^1_2.)

4. *Assigning colors to the remaining vertices of X.* Some vertices are not colored. They are now colored by pairwise different colors not already used.

5. *The end.* Obviously, $t \leqslant s$.

Lemma 3.1. *The maximal sieve through C_1 obtained in our construction $\mathscr{M}$ has a cardinality $\geqslant s - 1$, were s is the cardinality of a maximum sieve of $\mathscr{H}$. Hence $t \geqslant s - 1$.*

Proof. Let $\Sigma = \{A_1, A_2, \ldots, A_s\}$ be a maximum sieve of $\mathscr{H}$, and C_1 the starting $\mathscr{C}$-edge of the coloring procedure $\mathscr{M}$. The $\mathscr{C}$-edge C_1 intersects at most two $\mathscr{C}$-edges of Σ, say, A_1, A_2. Then $\{C_1\} \cup (\Sigma \setminus \{A_1, A_2\})$ is a maximal sieve of $\mathscr{H}$ of cardinality $\geqslant s - 1$. Therefore, procedure $\mathscr{M}$ generates a maximal sieve of size $\geqslant s - 1$. □

We say a $\mathscr{C}$-edge $C = \{y_1, \ldots, y_k\}$ lies between C_i and C_{i+1} if x^i_1 precedes y_1 and y_k precedes $x^{i+1}_{k_{i+1}}$.

Lemma 3.2. *Let C be a $\mathscr{C}$-edge lying between C_i and C_{i+1}, i.e., $C \subseteq C_i \cup C_{i+1} \setminus \{x^i_1, x^{i+1}_{k_{i+1}}\}$, $1 \leqslant i \leqslant t - 1$. Then C is properly colored by the colors of $x^i_{k_i-2}, x^i_{k_i-1}, x^i_{k_i}$, i.e., C contains two vertices of the same color from $C \cap C_i$.*

Proof. The $\mathscr{C}$-edge C_{i+1} has been chosen nearest to C_i. Hence the intersection $|C \cap C_i| \geqslant 2$ and if $|C \cap C_i| = 2$ then $C \cap C_i = \{x^i_{k_i-1}, x^i_{k_i}\}$ and $(x^i_{k_i-1}, x^i_{k_i}) \notin \mathscr{D}$. Thus C is properly colored by the colors of $x^i_{k_i-2}, x^i_{k_i-1}, x^i_{k_i}$, i.e., C contains two vertices of the same color of $C \cap C_i$. □

In the same way Lemma 3.3 can be proved.

Lemma 3.3. *Let C be a $\mathscr{C}$-edge lying between C_t and C_1, having a bad intersection $C \cap C_t$. Then C is properly colored by the colors of $x^t_{k_t-2}, x^t_{k_t-1}, x^t_{k_t}$, i.e., C contains two vertices of the same color of $C \cap C_t$.*

We change the names of the colors so that starting with the first vertex of color 1, say v, the colors $1, 2, \ldots, n - t$ are in this cyclic order on the host cycle. Let v be the first 1-colored vertex of C_1 and $v = v_1, v_2, \ldots, v_n$ be the vertices of X in this cyclic order. We partition the vertices (and the colors) into two classes (1) and (2) by the following

Partitioning procedure $\mathscr{P}$. Put v with color 1 in class (1). Let the vertices $v_0 = v, v_1, v_2, \ldots, v_{i-1}$ $(i \leqslant n)$ be already partitioned.

Let $v_{i-1} \in (j)$.

If $c(v_i) = c(v_{i-1})$ (hence $\{v_i, v_{i-1}\} \notin \mathscr{D}$) then put v_i in (j).

If $c(v_i) \neq c(v_{i-1})$ then put v_i in (1) if $j = 2$ and (2) if $j = 1$.

Obviously, if two vertices have the same color, then they are in the same class. Eventually, the vertex x^1_2 has two colors, and x^1_2 is with the first color 1 in (1) and with the second color in (2). Only x^1_2 can have two colors because the case that $C_1 = \{x_1, x_2, x_3\} \in \mathscr{C}$ with $\{x_1, x_2\}, \{x_2, x_3\} \in \mathscr{D}$ will be avoided by a special choice of C_1.

Next we consider the host path $w = (v = v_1, v_2, \ldots, v_{n-1}, v_n)$.

Lemma 3.4. *Let F_α denote the set of all vertices $p_1, \ldots, p_l$ in w of color α, $1 \leqslant \alpha \leqslant n - t$, which are in this cyclic order on w.*

The distance $d(p_i, p_{i+1}) \leqslant 2$ *for all* $1 \leqslant i \leqslant l-1$. *Moreover*,

(i) *if* $d(p_i, p_{i+1}) = 2$ *then* $\{q_i, p_{i+1}\} \in \mathscr{D}$, $1 \leqslant i \leqslant l-1$, *where* q_i *is the vertex between* p_i *and* p_{i+1}, *and*
(ii) *if* $d(p_i, p_{i+1}) = 1$ *then* $\{p_i, p_{i+1}\} \notin \mathscr{D}$, $1 \leqslant i \leqslant l-1$; $d(p_{i-1}, p_i) = 2$, $2 \leqslant i \leqslant l-1$; *and* $d(p_{i+1}, p_{i+2}) = 2$, $1 \leqslant i \leqslant l-2$.

Lemma 3.4 means that two consecutive vertices of w of the same color are not joint by a $\mathscr{D}$-edge of size 2, and no three vertices of w have the same color.

If x_2^1 is colored twice then the assertion of Lemma 3.4 is also true for the host path $(v_2, v_3, \ldots, v_n, v_1 = v)$, where $v = x_2^1$ is to be taken with the second color.

The main idea and aim of our further proofs is to correct the coloring obtained by $\mathscr{M}$ for some vertices lying on the host cycle between C_{t-2} and C_1 so that a proper coloring of the whole circular mixed hypergraph $\mathscr{H}$ is obtained.

4. Lower bounds for the upper chromatic number

Definition 4.1. Two different $\mathscr{C}$-edges $A = \{u, u^+, \ldots, u^{(a)}\}$ and $B = \{v, v^+, \ldots, v^{(b)}\}$ are said to have property Q if $A \cup B = X$ and $A \cap B$ induces two complete subgraphs K_l and K'_m of $(X, \mathscr{D})$ with l and m vertices, respectively, $1 \leqslant l, m \leqslant 2$, where $u, v^{(b)} \in K_l$ and $v, u^{(a)} \in K'_m$, and K_l and K'_m are separated by a vertex of A and a vertex of B.

Notice that circular mixed hypergraph $\mathscr{K}\mathscr{C}(2r+1; r+2, 2)$ always has two $\mathscr{C}$-edges with property Q.

Theorem 4.1. *Let* $r \geqslant 1$ *be an integer and* $n := 2r+1$. *Then* $s = s(\mathscr{K}\mathscr{C}(2r+1; r+2, 2)) = 1$ *and*

$$\bar{\chi}(\mathscr{K}\mathscr{C}(2r+1; r+2, 2)) = 2r-2 = n-3 = n-s-2 \quad \textit{for } r \geqslant 5, \textit{ and}$$

$$\bar{\chi}(\mathscr{K}\mathscr{C}(2r+1; r+2, 2)) = 2r-1 = n-2 = n-s-1 \quad \textit{for } 2 \leqslant r \leqslant 4.$$

Moreover, $\mathscr{K}\mathscr{C}(3; 3, 2) \cong \mathscr{F}_3$ *is not colorable*.

Proof. Obviously, $\mathscr{K}\mathscr{C}(3; 3, 2) \cong \mathscr{F}_3$ is not colorable ($r = 1$). It is evident $\bar{\chi}(\mathscr{K}\mathscr{C}(2r+1; r+2, 2)) \leqslant n-2$.

If $r = 2$ then $\mathscr{K}\mathscr{C}(5; 4, 2)$ has five vertices x_1, x_2, x_3, x_4, x_5 which are in this cyclic order on the host cycle. Coloring x_1, x_2, x_3, x_4, x_5 by 1, 2, 3, 1, 2, respectively, results in a strict coloring of $\mathscr{K}\mathscr{C}(5; 4, 2)$ by $3 = n-2$ colors.

If $3 \leqslant r \leqslant 4$ then $\mathscr{K}\mathscr{C}(2r+1; r+2, 2)$ has $2r+1$ vertices $x_1, x_2, x_3, x_4, x_5, x_6, x_7, \ldots$. If $r = 3$ (or $r = 4$) then x_1, x_4 and x_6, (or x_1, x_4 and x_7, respectively) are colored 1, and the other vertices receive the colors $2, 3, \ldots, n-2$. Thus a strict $(n-2)$-coloring of $\mathscr{K}\mathscr{C}(2r+1; r+2, 2)$ is obtained.

Next let $r \geqslant 5$. Firstly, we show that $\bar{\chi}(\mathscr{H}) \leqslant n-3$.

Suppose there is an integer $r \geqslant 5$ such that $\mathscr{H} := \mathscr{K}\mathscr{C}(2r+1; r+2, 2)$ has upper chromatic number $\bar{\chi}(\mathscr{H}) = n-2$. Let c denote a strict $(n-2)$-coloring of $\mathscr{H}$. We consider two cases.

Case 1. Let $\mathscr{H}$ have three vertices x, y, z of the same color, say 1. Then the other vertices of $\mathscr{H}$ are pairwise differently colored by $2, 3, \ldots, n-2$. Let $[x, y]$ be the xy-path of the host cycle not containing z. Correspondingly, $[y, z]$ and $[z, x]$ are defined. Let α, β and γ denote the lengths (edge numbers) of $[x, y]$, $[y, z]$, and $[z, x]$, respectively. Since $[x, y] \cup [y, z] \backslash \{x, z\}$ has no two vertices of the same color the length of $[x, y] \cup [y, z]$ is at most $r+2$, i.e., $\alpha + \beta \leqslant r+2$. Correspondingly, $\beta + \gamma \leqslant r+2$, and $\gamma + \alpha \leqslant r+2$. Summing up these three inequalities results in $2(2r+1) = 2n = 2(\alpha + \beta + \gamma) \leqslant 3(r+2)$, i.e., $r \leqslant 4$. This contradicts the hypothesis $r \geqslant 5$. Thus there is no strict $(n-2)$-coloring of $\mathscr{H}$ with three vertices having the same color.

Case 2. Let $\mathscr{H}$ have two pairs of vertices x_1, x_2 and y_1, y_2 with $1 := c(x_1) = c(x_2)$ and $2 := c(y_1) = c(y_2)$.

Subcase 2.1. Let x_1, y_1, x_2, y_2 be in this cyclic order on the host cycle. The vertices x_1, y_1, x_2, y_2 split the host cycle into four arcs $[x_1, y_1]$, $[y_1, x_2]$, $[x_2, y_2]$, and $[y_2, x_1]$. Let α, β, γ, and δ denote the lengths (edge numbers) of $[x_1, y_1]$, $[y_1, x_2]$, $[x_2, y_2]$, and $[y_2, x_1]$, respectively. Since $[x_1, y_1] \cup [y_1, x_2] \cup [x_2, y_2] \backslash \{x_1, y_2\}$ has no two vertices of equal color the length of $([x_1, y_1] \cup [y_1, x_2] \cup [x_2, y_2])$ is $\alpha + \beta + \gamma \leqslant (r + 2)$. Correspondingly, $\beta + \gamma + \delta \leqslant (r + 2)$, $\gamma + \delta + \alpha \leqslant (r + 2)$, and $\delta + \alpha + \beta \leqslant (r + 2)$. Summing up these four inequalities results in

$$3(2r + 1) = 3n = 3(\alpha + \beta + \gamma + \delta) \leqslant 4(r + 2) \quad \text{i.e.,} \quad r \leqslant 2.$$

This contradicts the hypothesis $r \geqslant 5$. Thus the proof of Theorem 4.1 in Subcase 2.1 is complete.

Subcase 2.2. Let x_1, x_2, y_1, y_2 be in this cyclic order on the host cycle. Let $\alpha, \beta, \gamma, \delta$ be the lenghts $[x_1, x_2]$, $[x_2, y_1]$, $[y_1, y_2]$, and $[y_2, x_1]$, respectively. Since $([x_1, x_2] \cup [x_2, y_1] \cup [y_1, y_2]) \backslash \{x_1, y_2\}$ and $([y_1, y_2] \cup [y_2, x_1] \cup [x_1, x_2]) \backslash \{y_1, x_2\}$ have no two vertices of the same color the lenghts of these paths are at most $r + 2$. Hence $\alpha + \beta + \gamma \leqslant (r + 2)$, and $\gamma + \delta + \alpha \leqslant (r + 2)$. Since $(X, \mathscr{D})$ is a cycle $\alpha \geqslant 2$ and $\gamma \geqslant 2$. Summing up these four inequalities results in

$(\alpha + \beta + \gamma) + (\gamma + \delta + \alpha) + 2 + 2 \leqslant (r + 2) + (r + 2) + \alpha + \gamma$.

Hence $2r + 1 = n \leqslant 2r$. This contradiction completes the proof of the assertion $\bar{\chi}(\mathscr{H}) \leqslant n - 3$.

In order to show that $\bar{\chi}(\mathscr{H}) \geqslant n - 3$ choose x_1, x_2, y_1, y_2 in Case 2.2 so that the lenghts of $[x_1, x_2]$, $[x_2, y_1]$, $[y_1, y_2]$, $[y_2, x_1]$ are $2, r - 1, 2, r - 2$, respectively. Let z_1 and z_2 denote the inner vertex of $[x_1, x_2]$ and $[y_1, y_2]$, respectively. We color x_1, x_2 by 1, y_1, y_2 by 2, z_1, z_2 by 3, and the other vertices by the colors $4, 5, \ldots, n - 3$. Thus a strict $(n - 3)$-coloring of $\mathscr{H}$ is obtained. This completes the proof ofTheorem 4.1. □

Theorem 4.2. *Let $\mathscr{H}$ be a reduced colorable circular mixed hypergraph with $n \geqslant 4$ vertices and sieve number s, $s \geqslant 1$. Let $\mathscr{H} \not\cong \mathscr{K}\mathscr{C}(2r+1; r+2, 2)$ for each integer $r \geqslant 1$. Let $\mathscr{H}$ contain two different $\mathscr{C}$-edges $A = \{u, u^+, \ldots, u^{(a)}\}$ and $B = \{v, v^+, \ldots, v^{(b)}\}$ which have property Q. Then $\mathscr{H}$ has a strict coloring by at least $n - s - 1$ colors.*

Proof. Notice that F_5 is not a subhypergraph of $\mathscr{H}$ since $\mathscr{H}$ is colorable. The condition $A \neq B$ implies $A \neq X \neq B$, and the complete subgraphs K_l and K'_m are separated by at least one vertex of A and at least one vertex of B.

Let $A = \{u, u^+, \ldots\}$, $B = \{v, v^+, \ldots\}$, and $|A| \geqslant |B|$.

We consider three cases $l = m = 2$, $l = m + 1 = 2$, and $l = m = 1$.

Case 1. Let $l = m = 2$.

The complete subgraphs are denoted by $K_2 = (\{u, u^+\}, \{(u, u^+)\})$ and $K'_2 = (\{v, v^+\}, \{(v, v^+)\})$. Obviously, $|A| \geqslant 5$ and $|B| \geqslant 5$. Since $\mathscr{H}$ is reduced each $\mathscr{C}$-edge $C \notin \{A, B\}$ contains $\{u^-, u, u^+, u^{++}\}$ or $\{v^-, v, v^+, v^{++}\}$. Hence $s \leqslant 2$. Coloring u^- and u^+ with 1, the vertices v^- and v^+ with 2, and giving all other vertices pairwise different colors from $3, \ldots, n - 2$ results in a strict $(n - 2)$-coloring of $\mathscr{H}$, where $n - 2 \geqslant n - s - 1$.

Case 2. $l = m = 1$.

The complete subgraphs are denoted by $K_1 = (\{u\}, \emptyset)$ and $K'_1 = (\{v\}, \emptyset)$.

Then $|A| \geqslant 3$, $|B| \geqslant 3$, and $n \geqslant 4$.

Subcase 2.1. $s \geqslant 2$.

Since $\mathscr{H}$ is reduced each $\mathscr{C}$-edge $C \notin \{A, B\}$ contains $\{u^-, u, u^+\}$ or $\{v^-, v, v^+\}$. Since both A and B cannot be in a maximum sieve (by definition of sieve), $s \leqslant 3$.

If $|A| \geqslant 4$, $|B| \geqslant 4$, then u^-, u^+ are colored 1, the vertices v^-, v^+ are colored 2, the vertices u, v are colored 3, and all other vertices are pairwise differently colored $4, \ldots, n - 3$.

If $|A| \geqslant 5$ and $|B| = 3$, then $v^-, v^+ = u^-, u^+$ are colored 1, the vertices u, v are colored 2, and all other vertices are pairwise differently colored $3, \ldots, n - 3$.

In both cases a strict $(n - 3)$-coloring of $\mathscr{H}$ is obtained, where $n - 3 \geqslant n - s - 1$.

If $|A| = 4$ and $|B| = 3$, then $n = 5$, $s \leqslant 3$, and either $\mathscr{F}_5$ is a partial subhypergraph of $\mathscr{H}$ and $\mathscr{H}$ is uncolorable or, evidently, the circular mixed hypergraph $\mathscr{H}$ has a strict 2- or 3-coloring, where $2 \geqslant n - s - 1$.

If $|A| = |B| = 3$, then $n = 4$, $s \leqslant 2$ and $\mathscr{H}$ has a strict 2-coloring, where $2 \geqslant n - s - 1$.

Subcase 2.2. $s = 1$.

We shall construct a strict $(n - 2)$-coloring because $n - 2 = n - s - 1$.

Let C be a $\mathscr{C}$-edge, $C \notin \{A, B\}$.

C contains $\{u^-, u, u^+\}$ or $\{v^-, v, v^+\}$, if $(u^-, u), (v^-, v) \notin \mathscr{D}$, or
C contains $\{u^{--}, u^-, u, u^+\}$ or $\{v^-, v, v^+\}$, if $(u^-, u) \in \mathscr{D}$, $(v^-, v) \notin \mathscr{D}$, or
C contains $\{u^-, u, u^+\}$ or $\{v^{--}, v^-, v, v^+\}$, if $(u^-, u) \notin \mathscr{D}$, $(v^-, v) \in \mathscr{D}$, or
C contains $\{u^{--}, u^-, u, u^+\}$ or $\{v^{--}, v^-, v, v^+\}$, if $(u^-, u), (v^-, v) \in \mathscr{D}$.

Note we use $s=1$, i.e., $\{u^-, u, u^+, \ldots\}$ is no $\mathscr{C}$-edge, if $(u^-, u) \in \mathscr{D}$, and $\{v^-, v, v^+, \ldots\}$ is no $\mathscr{C}$-edge, if $(v^-, v) \in \mathscr{D}$.

If $|A| \geqslant 4$, $|B| \geqslant 4$, then u^-, u or u^{--}, u are colored 1, if $(u^-, u) \notin \mathscr{D}$ or $(u^-, u) \in \mathscr{D}$, respectively. Correspondingly, v^-, v or v^{--}, v are colored 2, if $(v^-, v) \notin \mathscr{D}$ or $(v^-, v) \in \mathscr{D}$, respectively, and all other vertices are pairwise differently colored by $3, 4, \ldots, n-2$.

The obtained coloring is also proper, if $|A| \geqslant 4$, $|B| = 3$, and the additional condition $(u^-, u) \notin \mathscr{D}$ is fulfilled.

In these two cases a strict $(n-2)$-coloring is achieved.

Considering the host cycle in the opposite direction we again arrive at a strict $(n-2)$-coloring also in the case that $|A| \geqslant 4$, $|B| \geqslant 3$, and $(v, v^+) \notin \mathscr{D}$.

Next let $|A| \geqslant 5$, $|B| = 3$, and $(u^-, u), (v, v^+) \in \mathscr{D}$. Then $B = \{v, v^+ = u^-, u\}$ and $vv^+v^{++} = u^{--}u^-u$ is a path. Each $\mathscr{C}$-edge $C \notin \{A, B\}$ forms with B a sieve. Since $s = 1$, there is no such $\mathscr{C}$-edge C and $\mathscr{H} = (X, \{A, B\}, \mathscr{D})$. A strict $(n-1)$-coloring is obtained by coloring u, v with 1, and coloring the other at least three vertices pairwise differently with the colors 2, 3, 4.

Finally, let $|A| = |B| = 3$. Then $n = 4$ and $\mathscr{H}$ has a strict coloring with $n - s - 1 = 2$ colors.

Case 3. $l = m + 1 = 2$.

The complete subgraphs are denoted by $K_2 = (\{u, u^+\}, \{(u, u^+)\})$ and $K_1' = (\{v\}, \emptyset)$. Then $|A| \geqslant 4$, $|B| \geqslant 4$, and $n \geqslant 5$.

Subcase 3.1. $s \geqslant 2$.

Since $\mathscr{H}$ is reduced each $\mathscr{C}$-edge $C \notin \{A, B\}$ contains $\{u^-, u, u^+, u^{++}\}$ or $\{v^-, v, v^+\}$. Obviously, $s \leqslant 2$.

If $|A| \geqslant 5$ then u, u^{++}, v are colored 1, the vertices v^-, v^+ are colored 2, and all other vertices are pairwise differently colored $3, 4, \ldots, n-3$. Thus a strict $(n-3)$-coloring is obtained with $n - 3 \geqslant n - s - 1$, if $s \geqslant 2$.

If $|A| = |B| = 4$ then $n = 5$, and v^-, v^+ are colored 1, the vertices u, v are colored 2, and u^+ is colored 3. Thus a strict 3-coloring is obtained; $3 \geqslant n - s - 1$ (even in the case that $s = 1$).

Subcase 3.2. $s = 1$.

The circular mixed hypergraph $\mathscr{H}$ with $n \geqslant 5$ vertices does not contain $\mathscr{F}_n$ and $\mathscr{H}$ is colorable.

We shall construct a strict $(n-2)$-coloring because $n - 2 = n - s - 1$.

Subcase 3.2.1. $|B| = 4$.

Notice that $u^- = v^+$ and $|A| = |X| - 1$. Since $\mathscr{H}$ is reduced each $\mathscr{C}$-edge $C \notin \{A, B\}$ contains $\{u^-, u, u^+, u^{++}\}$ or $\{v^-, v, v^+\}$.

If $|A| = 4$, i.e., $u^{++} = v^-$ then $n = 5$. The vertices v^-, v^+ are colored 1, the vertices v, u are colored 2, and u^+ is colored $3 = n - s - 1$.

If $|A| \geqslant 5$, i.e., $u^{++} \neq v^-$ then v^-, v^+, u^+ are colored 1, and all other vertices are pairwise differently colored $2, 3, \ldots, n-2 = n - s - 1$.

Thus in both cases a strict coloring with $\geqslant n - s - 1$ colors is obtained.

Subcase 3.2.2. $|A| \geqslant |B| \geqslant 5$.

Denote $A_1 := A$ and $B_1 := B$. Each $\mathscr{C}$-edge $C \notin \{A_1, B_1\}$ contains $\{u^-, u, u^+, u^{++}\}$ or $\{v^-, v, v^+\}$.

If $(v, v^+) \notin \mathscr{D}$ then the vertices v, v^+ are colored 1, the vertices u, u^{++} are colored 2, and all other vertices are pairwise differently colored by $3, 4, \ldots, n-2 = n - s - 1$.

We claim

(*) If $(v, v^+) \in \mathscr{D}$ then each $\mathscr{C}$-edge $C \notin \{A_1, B_1\}$ contains $\{u^-, u, u^+, u^{++}\}$ or $\{v^-, v, v^+, v^{++}\}$ besides the case that $C = A_2 := \{u^+, u^{++}, \ldots, v^-, v, v^+\}$.

Proof of (*). Suppose there is a $\mathscr{C}$-edge $C \notin \{A_1, B_1\}$ such that $\{u^-, u, u^+, u^{++}\} \nsubseteq C$ and $\{v^-, v, v^+, v^{++}\} \nsubseteq C$. We know that $\{u^-, u, u^+, u^{++}\} \subseteq C$ or $\{v^-, v, v^+\} \subseteq C$. Since $\{u^-, u, u^+, u^{++}\} \nsubseteq C$ the set $\{v^-, v, v^+\} \subseteq C$. Hence $v^{++} \notin C$ and $C = \{v^+, v, v^-, \ldots\}$. The $\mathscr{C}$-edges B_1 and C form no sieve. Therefore, $u^+ \in C$, and $C = \{v^+, v, v^-, \ldots, u^+\}$ (Note that $\mathscr{H}$ is a reduced mixed hypergraph.) With $A_2 := C$ the proof of assertion (*) is complete.

Note: in the proof of (*) we have used s $= 1$.

If $A_2 \notin \mathscr{C}$ then the vertices v, v^{++} are colored 1, the vertices u, u^{++} are colored 2, and all other vertices are pairwise differently colored by $3, 4, \ldots, n-2 = n-s-1$.

The assertion (*) implies that in each case we have a strict $(n-2)$-coloring besides the case that $(v, v^+) \in \mathscr{D}$ and $A_2 \in \mathscr{C}$.

We repeat the coloring procedure with A_2 and B_1. A strict $(n-2)$-coloring is obtained besides the case that $(u^+, u^{++}) \in \mathscr{D}$ and $B_2 := \{v^+, v^{++}, \ldots, u, u^+, u^{++}\} \in \mathscr{D}$. In the latter case we repeat the coloring procedure with A_2 and B_2, and so on. By this method we either arrive at a strict $(n-2)$- coloring of $\mathscr{H}$ or $\mathscr{D} = \mathscr{D}_2$, i.e., $(X, \mathscr{D})$ is a cycle, and $\mathscr{C}$ contains a sequence $B_1, B_2, \ldots$ with $B_i = \{v^{i-1}, v^i, \ldots, u^{i-1}, u^i\}$ (Note that the same is true with $A_1, A_2, \ldots$.) Therefore, each sequence of $r+2 := |B_1|$ consecutive vertices is in $\mathscr{C}$, and no other subset of X is in $\mathscr{C}$. From $|A_1| = |B_1| = r+2$ it follows that $n = |X| = |A_1| + |B_1| - 3 = 2r+1$. Consequently, $\mathscr{H} \cong \mathscr{K}\mathscr{C}(2r+1; r+2, 2)$. This contradicts our hypothesis. Thus the proof of Theorem 4.2 is complete. □

Theorem 4.3. *Let $\mathscr{H}$ be a reduced circular mixed hypergraph with n vertices and sieve number s containing no two $\mathscr{C}$-edges of property Q. Let $\mathscr{H}$ satisfy one of the following conditions*:

(α) *$\mathscr{H}$ has a $\mathscr{C}$-edge of size $\geqslant 5$, or*

(β) *$\mathscr{D} \neq \mathscr{D}_2$, i.e., $(X, \mathscr{D})$ is no cycle, and $\mathscr{C} \subseteq \mathscr{C}_3 \cup \mathscr{C}_4$, i.e., each $\mathscr{C}$-edge contains precisely three or four vertices.*

Then $\mathscr{H}$ is colorable and $\bar{\chi}(\mathscr{H}) \geqslant n-s-1$.

Proof. If $\mathscr{H} = (X, \{X\}, \mathscr{D})$ then obviously $\bar{\chi}(\mathscr{H}) = n-1 = n-s$. In the following proof let $\mathscr{C} \neq \{X\}$, i.e., to each $\mathscr{C}$-edge C there is a vertex outside C.

Choice of C_1

Case (α). Let C_1 be a $\mathscr{C}$-edge of largest cardinality.

Case (β). We consider two subcases.

Case (β1). For all $\mathscr{C}$-edges $\{v, v^+, v^{++}\}$ and $\{w, w^+, w^{++}, w^{+++}\}$ it holds that the sets $\{v, v^+\}, \{v^+, v^{++}\} \in \mathscr{D}$ and $\{w, w^+\}, \{w^+, w^{++}\}, \{w^{++}, w^{+++}\} \in \mathscr{D}$. Let u, u^+ be two consecutive vertices of the host cycle with $\{u, u^+\} \notin \mathscr{D}$. Choose C_1 nearest to u^+. If $C_1 = \{x_1, x_2, \ldots\}$ then there is no $\mathscr{C}$-edge $C \neq C_1$ containing x_1.

Case (β2). There is a $\mathscr{C}$-edge containing a pair of consecutive vertices forming no $\mathscr{D}$-edge. Then by eventually changing the direction of the host cycle it is possible to choose a $\mathscr{C}$-edge $C_1 = \{x_1, x_2, x_3\}$ with $\{x_2, x_3\} \notin \mathscr{D}$ or a $\mathscr{C}$-edge $C_1 = \{x_1, x_2, x_3, x_4\}$ with $\{x_2, x_3\} \notin \mathscr{D}$ or $\{x_3, x_4\} \notin \mathscr{D}$.

In all cases let $C_1 = \{x_1, x_2, \ldots, x_{k_1}\}$, where $k_1 = |C_1|$. Later the notation is changed depending on the considered case.

Starting procedure $\mathscr{M}$ with the $\mathscr{C}$-edge C_1 we obtain an inclusionwise maximum sieve $\Sigma = \{C_1, C_2, \ldots, C_t\}$ through C_1 with a coloring c, where $C_1, \ldots, C_t$ are in this cyclic order on the host cycle. Lemma 3.1 implies $t \geqslant s-1$. It may be that the vertex x_2 (and only the vertex x_2) receives two colors by $\mathscr{M}$; each other vertex receives precisely one color by $\mathscr{M}$. If x_2 is colored precisely once or twice then c uses precisely $n-t$ or $n-t+1$ colors, respectively. We have to change c so that a strict coloring with $\geqslant n-t-1 \geqslant n-s-1$ colors is obtained, i.e., that the $\mathscr{C}$-edge and the $\mathscr{D}$-edge conditions are fulfilled.

By Lemma 3.2 all $\mathscr{C}$-edges between C_i and C_{i+1} are properly colored, $1 \leqslant i \leqslant t-1$; and by Lemma 3.3 all $\mathscr{C}$-edges between C_t and C_1 having a bad intersection with C_t are properly colored, too. So we have to take care for the $\mathscr{C}$-edges between C_t and C_1 having a good intersection with C_t and a bad intersection with C_1.

In the next part of the proof, we will use two methods for changing the coloring c. Firstly, we will identify two colors in the same color class (i), $i = 1$ or $i = 2$. Secondly, we will change some colors of C_1. By these recoloring methods properly colored $\mathscr{C}$-edges between C_i and C_{i+1} remain properly colored for $2 \leqslant i \leqslant t-1$.

Let $S := \{x_0, x_1, x_2\}$, if $\{x_1, x_2\} \notin \mathscr{D}$, and $S := \{x_0, x_1, x_2, x_3\}$, if $\{x_1, x_2\} \in \mathscr{D}$.

Let $T := \{x_{k_1-1}, x_{k_1}, x_{k_1+1}\}$, if $\{x_{k_1-1}, x_{k_1}\} \notin \mathscr{D}$, and $T := \{x_{k_1-2}, x_{k_1-1}, x_{k_1}, x_{k_1+1}\}$, if $\{x_{k_1-1}, x_{k_1}\} \in \mathscr{D}$.

Each $\mathscr{C}$-edge between C_t and C_1 with a bad intersection with C_1 contains S, and each $\mathscr{C}$-edge between C_1 and C_2 with a bad intersection with C_1 contains T. Thus the $\mathscr{C}$-edge condition is satisfied for the new coloring if each of the three sets C_1, S, and T has two vertices of the same color. By our recoloring method properly colored $\mathscr{D}$-edges outside C_1 remain properly colored. Thus the $\mathscr{D}$-edge condition is satisfied for the new coloring if any two consecutive vertices of $\{x_{-1}, x_0, x_1, \ldots, x_{k_1}, x_{k_1+1}, x_{k_1+2}\}$ are colored differently, if they are joint by a $\mathscr{D}$-edge of size two, and any three consecutive vertices of this set have not the same color.

Many cases have to be considered. This case analysis is organized in (A), (B), (C), (D), and (E) as follows:

(i) $|C_1| \geqslant 6$, $C_1 = \{x_1, x_2, \ldots\}$. The proof is in (B).
(ii) $|C_1| = 5$, $C_1 = \{x_1, x_2, x_3, x_4, x_5\}$.
  (a) If $\{x_4, x_5\} \notin \mathscr{D}$ then the proof is in (B).
  (b) If $\{x_4, x_5\} \in \mathscr{D}$ then the proof is in (D).
(iii) $|C_1| = 4$, $C_1 = \{x_1, x_2, x_3, x_4\}$.
  (a) If $\{x_1, x_2\}, \{x_3, x_4\} \notin \mathscr{D}$ then the proof is in (B).
  (b) If $\{x_1, x_2\} \in \mathscr{D}$, $\{x_3, x_4\} \notin \mathscr{D}$ then the proof is in (C).
  (c) If $\{x_1, x_2\}, \{x_3, x_4\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$ and x_2 is colored only once by $\mathscr{M}$ then the proof is in (D).
  (d) If $\{x_1, x_2\}, \{x_3, x_4\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$ and x_2 is colored twice by $\mathscr{M}$ then the proof is in (E).
  (e) If $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\} \in \mathscr{D}$then the proof is in (A).
(iv) $|C_1| = 3$, $C_1 = \{x_1, x_2, x_3\}$.
  (a) If $\{x_1, x_2\}, \{x_2, x_3\} \notin \mathscr{D}$ then the proof is in (C).
  (b) If $\{x_1, x_2\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$ and x_2 is colored only once by $\mathscr{M}$ then the proof is in (C).
  (c) If $\{x_1, x_2\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$ and x_2 is colored twice by $\mathscr{M}$ then the proof is in (E).
  (d) If $\{x_1, x_2\}, \{x_2, x_3\} \in \mathscr{D}$ then the proof is in (A).

Case A (iii(e), iv(d)). $\mathscr{D} \neq \mathscr{D}_2, \mathscr{C} \subseteq \mathscr{C}_3 \cup \mathscr{C}_4$ and for each $\mathscr{C}$-edge $\{v, v^+, v^{++}\}$ the 2-sets $\{v, v^+\}, \{v^+, v^{++}\} \in \mathscr{D}$ and for each $\mathscr{C}$-edge $\{w, w^+, w^{++}, w^{+++}\}$ the 2-sets $\{w, w^+\}$, $\{w^+, w^{++}\}$, $\{w^{++}, w^{+++}\} \in \mathscr{D}$ (This is Case (β1).)

Let $\{u, u^+\} \notin \mathscr{D}$ (as it has been chosen at the beginning of the proof). Since C_1 is nearest to u^+ no $\mathscr{C}$-edge $C \neq C_1$ contains x_1.

By $\mathscr{M}$ the $\mathscr{C}$-edge condition is fulfilled, and besides perhaps u, u^+ no two consecutive vertices have the same color. Hence no three consecutive vertices have the same color, and the $\mathscr{D}$-edge condition is satisfied, too. Thus a strict $(n - t)$-coloring is obtained.

Case B (i, ii(a), iii(a)). $|C_1| \geqslant 6$ or $C_1 = \{x_1, x_2, x_3, x_4, x_5\}$ with $\{x_4, x_5\} \notin \mathscr{D}$ or $C_1 = \{x_1, x_2, x_3, x_4\}$ with $\{x_1, x_2\}$, $\{x_3, x_4\} \notin \mathscr{D}$.

If $|C_1| \geqslant 6$ or $C_1 = \{x_1, x_2, x_3, x_4, x_5\}$ with $\{x_4, x_5\} \notin \mathscr{D}$ then x_3 has a *non-repeated* color, i.e., no vertex of $X \backslash \{x_3\}$ has the color $c(x_3)$ of x_3. By $\mathscr{M}$ the $\mathscr{D}$-edge condition is fulfilled. If the set $\{x_0, x_1, x_2\}$ has two vertices of the same color then the $\mathscr{C}$-edge condition is satisfied, too, and $\mathscr{H}$ has a strict $(n - t)$-coloring; otherwise recolor x_2 by the color $c(x_0)$ and $\mathscr{H}$ has a strict $(n - t - 1)$-coloring.

If $C_1 = \{x_1, x_2, x_3, x_4, \}$ with $\{x_1, x_2\}, \{x_3, x_4\} \notin \mathscr{D}$ then x_2 has the *non-repeated* color. By $\mathscr{M}$ the $\mathscr{D}$-edge condition is fulfilled. If the set $\{x_0, x_1, x_2\}$ has two vertices of the same color then the $\mathscr{C}$-edge condition is satisfied, too, and $\mathscr{H}$ has a strict $(n - t)$-coloring; otherwise recolor x_2 by the color $c(x_0)$ or $c(x_1)$, if $c(x_0) \neq c(x_3) = 1$ or $c(x_0) = c(x_3) = 1$, respectively, and a strict $(n - t - 1)$-coloring is obtained.

Case C (iii(b), iv(a), iv(b)). $C_1 = \{x_1, x_2, x_3\}$ with $\{x_1, x_2\}, \{x_2, x_3\} \notin \mathscr{D}$, or $C_1 = \{x_1, x_2, x_3\}$ with $\{x_1, x_2\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$ and x_2 is colored only once by $\mathscr{M}$, or $C_1 = \{x_0, x_1, x_2, x_3\}$ with $\{x_0, x_1\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$. In all these cases x_2 is colored only once by $\mathscr{M}$. By $\mathscr{M}$ the colors $c(x_2) = c(x_3) = 1$ and $c(x_4) \neq 1$, say $c(x_4) = 2$.

Each $\mathscr{C}$-edge C between C_t and C_1 with a bad intersection $C \cap C_1$ with C_1 contains $\{x_0, x_1, x_2\}$, if $|C_1| = 3$, and $\{x_{-1}, x_0, x_1, x_2\}$, if $|C_1| = 4$.

Subcase C1. If $x_0, x_1 \in (i)$, $i = 1$, or $i = 2$, then $\mathscr{M}$ implies $c(x_0) = c(x_1)$ (and $|C_1| = 3$).

If $c(x_0) = c(x_1) \neq 1$ then c is a strict $(n - t)$-coloring.

If $c(x_0) = c(x_1) = 1$ then recoloring x_2 by 2 results in a strict $(n - t)$-coloring. (Note that x_2 changes from (1) to (2).)

Subcase C2. If $x_0 \in (1)$ and $x_1 \in (2)$ then replacing the color $c(x_0)$ by 1 results in a strict coloring with $\geqslant n - t - 1$ colors.

Subcase C3. Let $x_0 \in (2)$ and $x_1 \in (1)$.

If $|C_1| = 4$ then we replace the color of all vertices of color $c(x_0)$ and the color of the vertex x_2 by 2, and obtain a strict coloring by $\geqslant n - t - 1$ colors.

If $C_1 = \{x_1, x_2, x_3\}$, and $\{x_1, x_2\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$ then there is no $\mathscr{C}$-edge between C_t and C_1 having a bad intersection $C_t \cap C_1$ with C_1, and $\{x_0, x_1, x_2\}$ can be multicolored in a strict coloring of $\mathscr{H}$. If $c(x_1) \neq 1$ then c is a strict $(n - t)$-coloring. If $c(x_0) = 1$ then recoloring $c(x_2)$ by 2 results in a strict $(n - t)$-coloring.

Next let $C_1 = \{x_1, x_2, x_3\}$ and $\{x_1, x_2\}, \{x_2, x_3\} \notin \mathscr{D}$. We replace the color of all vertices of color $c(x_1)$ by 1 and recolor all vertices of color $c(x_0)$ and the vertex x_2 by 2. Thus a strict coloring with $\geqslant n - t - 2$ colors is obtained.

Precisely $n - t - 2$ colors are achieved if $c(x_0) \neq 2$, say $c(x_0) = 4$, and $c(x_1) \neq 1$, say $c(x_1) = 3$.

Our method $\mathscr{M}$ implies: on the x_2x_1-path of the host cycle there are not at the same time two consecutive vertices of the colors 1 and 4 and two consecutive vertices ofthe colors 2 and 3 (see Lemma 3.4). If there are no two consecutive vertices of the colors 1 and 4 we recolor all vertices of color 4 by 1. If there are no two consecutive vertices of the colors 2 and 3 we recolor all vertices of color 3 and the vertex x_2 by 2. Thus we obtain a strict $(n - t - 1)$-coloring of $\mathscr{H}$.

Case D (ii(b), iii(c)). $C_1 = \{x_1, x_2, x_3, x_4\}$ with $\{x_1, x_2\}, \{x_3, x_4\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$ and x_2 is colored only once by $\mathscr{M}$, or $C_1 = \{x_0, x_1, x_2, x_3, x_4\}$ with $\{x_0, x_1\}, \{x_3, x_4\} \in \mathscr{D}$. In the latter case it can be assumed $\{x_0, x_1\} \in \mathscr{D}$; otherwise by changing the direction of the host cycle we arrive at Case (A).

By $\mathscr{M}$ the colors $c(x_2) = c(x_4) = 1$, and $c(x_3) \neq 1$, say $c(x_3) = 2$.

If $x_5 \in (1)$ then $c(x_4) = c(x_5) = 1$ and $\{x_4, x_5\} \notin \mathscr{D}$. By $\mathscr{M}$ the set $\{x_3, x_4, x_5\} \in \mathscr{C}$ with $\{x_4, x_5\} \notin \mathscr{D}$. Again starting $\mathscr{M}$ with the $\mathscr{C}$-edge $C_1' := \{x_3, x_4, x_5\}$ we arrive at Case (C) or Case (E).

Suppose $x_5 \in (2)$. Each $\mathscr{C}$-edge C between C_t and C_1 with a bad intersection $C \cap C_1$ with C_1 contains $\{x_0, x_1, x_2, x_3\}$, if $|C_1| = 4$, and $\{x_{-1}, x_0, x_1, x_2\}$, if $|C_1| = 5$

Subcase D1. If $x_0, x_1 \in (i)$ for $i = 1$ or $i = 2$ then $\mathscr{M}$ implies $c(x_0) = c(x_1)$, $\{x_0, x_1\} \notin \mathscr{D}$, and $|C_1| = 4$. Thus c is a strict $(n - t)$-coloring besides the case that $c(x_0) = c(x_1) = 1$. In this case recoloring x_2 by 2 results in a strict $(n - t - 1)$-coloring of $\mathscr{H}$.

Subcase D2. If $x_0 \in (1)$ and $x_1 \in (2)$ then recoloring all vertices of color $c(x_0)$ by 1 results in a strict coloring with $\geqslant n - t - 1$ colors.

Subcase D3. $x_0 \in (2)$ and $x_1 \in (1)$.

If $|C_1| = 4$ then recoloring all vertices of color $c(x_0)$ by 2 results in a strict coloring with $\geqslant n - t - 1$ colors besides the case that $c(x_1) = 1$. In the latter case also replacing the color 1 of x_2 by 2 results in a strict coloring with $\geqslant n - t - 1$ colors.

Next let $|C_1| = 5$.

If $c(x_1) = 1$ then by $\mathscr{M}$ the color $c(x_{-1}) = c(x_1) = 1$ and $x_0 \in (2)$. Recoloring all vertices of color $c(x_5)$ by 2 and giving x_2 a completely new color results in a strict coloring by $\geqslant n - t$ colors.

Next let $c(x_1) \neq 1$.

If $\{x_{-1}, x_0, x_1, x_2\}$ contains two vertices of the same color then c is a strict $(n - t)$-coloring. If $\{x_{-1}, x_0, x_1, x_2\}$ is multicolored then recoloring x_1 by $c(x_{-1})$ results in a strict $(n - t - 1)$-coloring. Note $c(x_{-1}) \neq 1$. Thus again a strict $(n - t - 1)$-coloring is obtained.

Case E (iii(d), iv(c)). x_2 is colored twice by $\mathscr{M}$, i.e., in the course of $\mathscr{M}$ the vertex x_2 is firstly colored 1 and secondly colored β, $\beta \neq 1$.

This can only occur in the case that $C_1 = \{x_1, x_2, x_3\} \in \mathscr{C}$, $\{x_1, x_2\} \in \mathscr{D}$, $\{x_2, x_3\} \notin \mathscr{D}$, or $C_1 = \{x_1, x_2, x_3, x_4\} \in \mathscr{C}$, $\{x_1, x_2\}, \{x_3, x_4\} \in \mathscr{D}$, and $\{x_2, x_3\} \notin \mathscr{D}$. In this coloring precisely $n - t + 1$ colors are used, because at the vertex x_2 two colors are repeated, namely, 1 and β.

The procedure $\mathscr{M}$ implies $c(x_3) = 1$, if $|C_1| = 3$, and $c(x_4) = 1$, if $|C_1| = 4$. Moreover, $c(x_0) = \beta$ and $\alpha := c(x_1) \neq \beta$. By $\mathscr{M}$ the color $c(x_4) \neq 1$, say $c(x_4) = 2$, if $|C_1| = 3$, and the color $c(x_3) \neq 1$, say $c(x_3) = 2$, if $|C_1| = 4$.

Subcase E1. $\beta \in (1)$. Hence $\alpha \in (2)$.

Recoloring x_2 and all other vertices of color β by 1 results in a strict coloring with $\geqslant n - t$ colors.

Subcase E2. $\beta \in (2)$. Hence $\alpha \in (1)$.

Recoloring x_2 and all other vertices of color β by 2 and giving all vertices of color α the color 1 results in a proper coloring of $\mathscr{H}$. The colors α and β can disappear. Hence at least $(n - t + 1) - 2 = n - t - 1$ colors are used. □

Theorem 4.4. *Let $\mathscr{H}$ be a reduced circular mixed hypergraph with n vertices, sieve number s, $\mathscr{C} \subseteq \mathscr{C}_3 \cup \mathscr{C}_4$, and $\mathscr{D} = \mathscr{D}_2$. Let $\mathscr{H}$ do not contain two $\mathscr{C}$-edges of property Q. Then either n is odd and $\mathscr{F}_n$ is a subhypergraph of $\mathscr{H}$ or $\mathscr{H}$ is colorable and $\bar{\chi}(\mathscr{H}) \geqslant n - s - 1$.*

Proof. If $\mathscr{C}$ contains no quadruple then the assertion follows from Theorems 2.3 and 2.4.

Let $x_0, x_1, \ldots, x_n = x_0$ be the vertices of $\mathscr{H}$ in this cyclic order on the host cycle such that $C_1 := \{x_0, x_1, x_2, x_3\}$ is a quadruple.

Starting procedure $\mathscr{M}$ with the $\mathscr{C}$-edge C_1 we obtain a maximum (by inclusion) sieve $\Sigma = \{C_1, C_2, \ldots, C_t\}$ through C_1 with a coloring c. Lemma 3.1 implies $t \geqslant s - 1$. It may be that the vertex x_1 and only the vertex x_1 receives two colors by $\mathscr{M}$; otherwise each vertex receives precisely one color by $\mathscr{M}$. We have to change c so that a strict coloring with $\geqslant n - t - 1 \geqslant n - s - 1$ colors is obtained, i.e., that the $\mathscr{C}$-edge and $\mathscr{D}$-edge conditions are fulfilled. By Lemma 3.2, all $\mathscr{C}$-edges between C_i and C_{i+1} are properly colored, $1 \leqslant i \leqslant t - 1$; and by Lemma 3.3 all $\mathscr{C}$-edges between C_t and C_1 having a bad intersection with C_t are properly colored, too. So we have to take care for the $\mathscr{C}$-edges between C_t and C_1 having a good intersection with C_t and a bad intersection with C_1. We have also to take care for the case that three consecutive vertices from $\{x_{-1}, x_0, \ldots, x_3\}$ have the same color.

We will use two methods for changing the coloring c. Firstly, we will identify two colors in the same color class (i), $i = 1$ or $i = 2$. Secondly, we will change some colors of C_1. By these recoloring methods properly colored $\mathscr{C}$-edges between C_i and C_{i+1} remain properly colored for $2 \leqslant i \leqslant t - 1$.

Let $S := \{x_0, x_1, x_2, x_3\}$, notice that $\{x_1, x_2\} \in \mathscr{D}$.

Let $T := \{x_{k_1-2}, x_{k_1-1}, x_{k_1}, x_{k_1+1}\}$, notice that $\{x_{k_1-1}, x_{k_1}\} \in \mathscr{D}$.

Each $\mathscr{C}$-edge between C_t and C_1 with a bad intersection with C_1 contains S, and each $\mathscr{C}$-edge between C_1 and C_2 with a bad intersection with C_1 contains T. Thus the $\mathscr{C}$-edge condition is satisfied for the new coloring if each of the three sets C_1, S, and T has two vertices of the same color. By our recoloring method properly colored $\mathscr{D}$-edges outside C_1 remain properly colored. Thus the $\mathscr{D}$-edge condition is satisfied for the new coloring if any two consecutive vertices of $\{x_{-1}, x_0, x_1, \ldots, x_{k_1}, x_{k_1+1}, x_{k_1+2}\}$ are colored differently, and any three consecutive vertices of this set have not the same color.

The procedure $\mathscr{M}$ implies $c(x_1) = c(x_3) = 1$ and $c(x_2) = 2$.

Since $(X, \mathscr{D})$ is a cycle no two consecutive vertices get the same color by $\mathscr{M}$. Hence $(1) = \{x_1, x_3, x_5, \ldots\}$ and $(2) = \{x_2, x_4, x_6, \ldots\}$.

We consider two cases.

Case A. The vertex x_1 is colored only once by $\mathscr{M}$.

If n is even then $x_0 \in (2)$ and recoloring all vertices of color $c(x_0)$ by 2 results in a strict coloring by $\geqslant n - t - 1$ colors.

Next let n be an odd integer. Then $x_0 \in (1)$.

If $c(x_0) \neq 1$ the recoloring all vertices of color $c(x_{-1})$ by 2 results in a strict coloring by $\geqslant n - t - 1$ colors.

If $c(x_0) = c(x_1) = 1$ then recoloring all vertices of color $c(x_4)$ by 2 and all vertices of color $c(x_{-1})$ by 2 and giving x_1 a completely new color γ results in a strict coloring by $\geqslant n - t - 1$ colors.

Case B. The vertex x_1 be colored twice by $\mathscr{M}$, i.e., in the course of $\mathscr{M}$ the vertex x_1 is firstly colored 1 and secondly colored β, $\beta \neq 1$.

The color $c(x_{-1}) = \beta$. We color the vertex x_1 by β and again denote the obtained coloring with c. Obviously, c uses $n - t + 1$ colors.

If n is even then $\beta \in (1)$. Recoloring all vertices of color β by 1 results in a strict coloring by $\geqslant n - t$ colors.

Next let n be an odd integer. Then $\beta \in (2)$.

Subcase B1. $\beta \neq 2$.

If $c(x_4) = \beta$ then recoloring all vertices of color $c(x_0)$ by 1 results in a strict coloring by $\geqslant n - t$ colors.

If $c(x_4) \neq \beta$ then recoloring all vertices of color $c(x_0)$ by 1 and all vertices of color $c(x_4)$ by 2 results in a strict coloring by $\geqslant n - t - 1$ colors.

Subcase B2. $\beta = 2$.

Then $c(x_2) = c(x_4) = \cdots = c(x_{-1}) = c(x_1) = 2$. In each of these vertices, besides x_2, ends a $\mathscr{C}$-edge of Σ. Let Σ' denote the set of these $\mathscr{C}$-edges. Obviously, they have sizes 3 and 4. If all $\mathscr{C}$-edges of Σ' have size 3 then Σ' and $(X, \mathscr{D})$ form an $\mathscr{F}_n$ and $\mathscr{H}$ is not colorable. Suppose that one $\mathscr{C}$-edge of Σ' has size 4. Let $C' := \{x_{2r-1}, x_{2r}, x_{2r+1}, x_{2r+2}\}$, $1 \leqslant r \leqslant (n-1)/2$, be the $\mathscr{C}$-edge of Σ' of size 4, which is last met by walking on the host cycle from x_1 to $x_n = x_0$.

If $r = 1$ then $C' = \{x_1, x_2, x_3, x_4\}$ has the bad intersection $\{x_1, x_2, x_3\}$ with $C_1 = \{x_0, x_1, x_2, x_3\}$. Since $C_1, C' \in \Sigma$, the $\mathscr{C}$-edge $C' \neq \{x_1, x_2, x_3, x_4\}$, and $r \geqslant 2$. Hence $\{x_{2r+2}, x_{2r+3}, x_{2r+4}\}, \ldots, \{x_{-1}, x_0, x_1\} \in \Sigma'$, $2 \leqslant r \leqslant (n - 1)/2$.

If $r = (n - 1)/2$, i.e., $2r + 1 = n + 1$ and $C' = \{x_{-2}, x_{-1}, x_0, x_1\}$, then we recolor all vertices of the colors $c(x_{-2})$ and $c(x_0)$ by 1. The vertex x_1 gets a completely new color, say α. Thus $\mathscr{H}$ has a strict coloring by $\geqslant n - t$ colors.

Next let $2 \leqslant r \leqslant (n - 1)/2 - 1 = (n - 3)/2$. Obviously, $C_t = \{x_{-1}, x_0, x_1\}$.

We recolor all vertices of colors $c(x_{2r+1})$ and $c(x_{2r+3})$ by the color $c(x_{2r-1})$. By this change properly colored $\mathscr{C}$-edges are again properly colored. Now C' contains the vertices x_{2r-1}, x_{2r+1} of equal color, and each $\mathscr{C}$-edge C between C' and the following $\mathscr{C}$-edge C'' of Σ has the two vertices x_{2r+1}, x_{2r+3} of the same color. Hence after replacing the color 2 of the vertices $x_{2r+2}, x_{2r+4}, \ldots, x_{n+1} = x_1$ by a completely new color α the $\mathscr{C}$-edges C and C' are properly colored.

Finally, replacing the color $c(x_0)$ by 1 causes a proper coloring of C_1. Since $C_t = \{x_{-1}, x_0, x_1\}$ there is no $\mathscr{C}$-edge between C_t and C_1. Thus $\mathscr{H}$ has a strict coloring by $\geqslant (n-t+1) - 2 + 1 - 1 = n - t - 1$ colors. □

5. Conclusion

In this section we formulate the main conclusion of the paper.

Theorem 5.1. *If $\mathscr{H} = (X, \mathscr{C}, \mathscr{D})$ is a reduced colorable circular mixed hypergraph with n vertices and sieve number s, then*

$$n - s - 2 \leqslant \bar{\chi}(\mathscr{H}) \leqslant n - s + 2,$$

and these bounds are sharp. Moreover, there exist circular mixed hypergraphs with the upper chromatic number equal to any intermediate value within the bounds above.

Proof. The upper bound follows from Theorem 2.1.

For the lower bound, if $\mathscr{H}$ has two $\mathscr{C}$-edges with property Q, then apply Theorems 4.1 and 4.2. If $\mathscr{H}$ does not have two $\mathscr{C}$-edges with property Q, then apply Theorem 4.3 (if $\mathscr{H}$ has a $\mathscr{C}$-edge of size $\geqslant 5$, no restrictions on $\mathscr{D}$, or, all $\mathscr{C}$-edges have size 3 or 4 and $(X, \mathscr{D})$ is not a cycle), and Theorem 4.4 (if all $\mathscr{C}$-edges have size 3 or 4 and $(X, \mathscr{D})$ is a cycle).

The last statement follows from Theorems 2.2, 2.3, 2.4, 2.5 and 4.1. □

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